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Opinion Dynamics and Stubbornness via Multi-Population Mean-Field Games

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Abstract This paper studies opinion dynamics for a set of heterogeneous populations of individuals pursuing two conflicting goals: to seek consensus and to be coherent with their initial opinions. The multi-population game under investigation is characterized by (i) rational agents who behave strategically, (ii) heterogeneous populations, (iii) opinions evolving in response to local in-

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teractions. The main contribution of this paper is to encompass all of these aspects under the unified framework of mean-field game theory.

We show that, assuming initial Gaussian density functions and affine control policies, the Fokker-Planck-Kolmogorov equation preserves Gaussianity over time. This fact is then used to explicitly derive expressions for the optimal control strategies when the players are myopic. We then explore consensus formation depending on the stubbornness of the involved populations: we identify conditions that lead to some elementary patterns, such as consensus, polarization or plurality of opinions.

Finally, under the baseline example of the presence of a stubborn population and a most gregarious one, we study the behavior of the model with a finite number of players, describing the dynamics of the average opinion, which is now a stochastic process. We also provide numerical simulations to show how the parameters impact the equilibrium formation.

Keywords Opinion dynamics, · Consensus · Heterogeneous populations · Stubbornness · Mean-Field Games

Mathematics Subject Classification (2000) 91A13 · 91B10 · 91B69 · 91D30

1 Introduction

Opinion dynamics describe the time evolution of the opinions in a large population of players as the result of repeated interactions among each other over a social network (see, e.g., [1, Sect. III] and [2]). Agents belonging to a social network are continuously bombed by information and their opinions change in time because of *emulation* or *herd behavior* [1,3–7].

In the literature, we speak about *consensus* whenever opinions converge to a unique value, of *polarization* when the consensus values are multiple but few in number, or even of *plurality* if the consensus values are multiple and numerous. A main cause of plurality is represented by "bounded confidence" (see [5]) whereby players do not take into account the opinion of agents whose beliefs are too different. A similar phenomenon occurs in the presence of *stubborn players* (see [8], [9]) such as leaders of political parties or media sources. These are players who do not feedback their neighbors' opinions while at the same time try to influence the consensus dynamics.

Our aim is to build a tractable model capable of discussing consensus formation in social networks of heterogeneous agents that interact strategically. More specifically, we study opinion dynamics for a set of heterogeneous populations of individuals pursuing two conflicting goals: social recognition (due to imitation, herding); confirmation of initial opinions (stubbornness). As previously stated, populations are characterized by different levels of stubbornness. There are *hard core* stubborn populations in which the players ignore inputs from neighboring populations; *most gregarious* populations in which the players are extremely susceptible to the inputs received from their neighboring populations; *partially stubborn* populations in which the players show a mixed behavior. The final goal is to determine under which conditions on the characteristics of the agents and of the social network, the individuals reach consensus on a single or a plurality of opinions.

Our analysis takes advantage of the theory of mean-field games where all the main ingredients (strategic interactions, heterogeneity, network structure) can be considered, maintaining tractability. As a matter of fact, interactions among economic agents is a key concept in the theory of mean-field games (see [10–13]). The underlying idea is that a large number of indistinguishable players interact so that the strategy of a single player is influenced by the distribution of the other players. Explicit solutions are available in [14] for the linear-quadratic case, and are extended to more general cases in [15]. However, in [16], it is pointed out that determining the solutions of a mean-field game is often impractical from a computational point of view; moreover, it does not naturally correspond to any reasonable dynamic process that agents are likely to follow in practice. For this reason, in [16] it is assumed that the agents implement myopic learning dynamics.

Our main contribution is to provide an explicit expression for the optimal control under the assumption that the players optimize myopically (i.e., assuming that moments of populations distributions are constant in time). The obtained control, once entered into the players' dynamics, sheds light on the elementary behavioral patterns of the populations as a whole. In particular, we group such elementary behaviors into three cases, i) multiple hard core populations, ii) all populations most gregarious, and iii) one population hard core and the remaining ones most gregarious. Results in this context are proved by making use of graph theory. In doing this, we generalize [17] to prove that affine control policies preserve the Gaussianity of populations' distributions under specific cost structures. This is done by first formulating the problem as a game with an infinite number of players and using a mean-field approach. The game involves two partial differential equations (PDEs). The first PDE is the Hamilton-Jacobi-Bellman equation, which is coupled to a second PDE, the Fokker-Planck-Kolmogorov equation, which captures the time evolution of the density in the space of opinions [13, 18]. As a further result, we study the case with a finite number of players on a guideline example with two populations, a hard core one and a most gregarious one. The average opinion, computed as the sample average over a finite number of players, is a stochastic process; we provide bounds on its first and second moments using tools borrowed from stochastic stability theory.

This paper is organized as follows. In Section 2, we set-up the optimization problem for a large network of interacting agents, discussing affine Gaussian Distribution Preserving Strategies. In Section 3 we solve the optimization problem in the case of myopic agents. As an illustrative example, in Section 4 we discuss the case of a hard core stubborn population vs. a most gregarious one. In particular, we show how the finite dimensional model behaves in this case, also by means of numerical simulations. Section 5 contains conclusions. In Appendix A, we show in details how to interpret the optimization problem in terms of a mean-field game.

2 A Network of Interacting Agents

Consider a set $I = \{1, \ldots, n\}$ of n distinct and interconnected populations. To model interconnections we use a graph $G = (I, E)$ where the node set is the set of populations I, and E collects all the weighted edges. More precisely, arc (i, j) belongs to E if the corresponding weight $\nu_{ij} > 0$. A positive scalar weight ν_{ij} represents the relative importance that population i assigns to population j. For instance ν_{ij} could be computed as the number of players of population j over the total number of players of all the neighbor populations. Hereinafter, we assume that G is *strongly connected* unless differently specified. This means that there is a directed path connecting any two arbitrary nodes of the graph.

There are N agents populating the social network. We denote by $i(k) \in I$, the label of the population to which k belongs. Each agent $k = 1, \ldots, N$ is represented by a (real) state $x^k(t)$ evolving in time. In particular, the state of the generic player k evolves according to

$$dx^k(t) = u^k(t)dt + \xi_{i(k)}dW^k(t), \quad x^k(0) = x_0^k \tag{1}$$

where $(u^k(t))_{t\geq 0} \in \mathcal{U}$ is a suitable control (to be specified later); $(\xi_i)_{i\in I}$ is a vector of population dependent volatility terms; $(W^k(t))_{t\geq 0}$, $k = 1, \ldots, N$ are independent standard Brownian motions leaving in a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$; Brownian terms account for misspecifications or generic stochastic disturbances affecting the evolution of the opinion of the single agents.

Since $x^k(t)$ is stochastic, the distribution of the populations' opinion will change in time. We denote by $m_i(x, t)$ the probabilistic density function of the state of players of population $i \in I$ at time t; we write $m(x,t)$ or simply $m(t)$ in place of $[m_i(x,t)]_{i\in I}$ and we denote by $\mu_i(t)$ the mean value of $m_i(x,t)$.

Individuals are *rational*: they adjust their opinion by solving an *optimal control problem.* As previously stated, there are two opposite driving forces. From one side, agents tend to align with more recognized opinions, from the other side, they are willing to be coherent with initial ideas. Moreover, any change in the strategy has a cost. As it will become clearer later, in deciding their strategy, agents act strategically, in the sense that they have to predict the evolution of others' opinions in order to take their *optimal* strategy. In this sense, they play a continuous time game among each other: in Appendix A, we show how to rewrite the problem at hand as a multi-population mean-field game in the sense of [13].

We define the running cost c^k for agent k belonging to population $i(k) = i$ as

$$\begin{aligned} &c^k(x^k(t), u^k(t), m(t)) = \\ &\qquad (1-\alpha_i)\sum_{j\in I}\big(\nu_{ij}\ln(m_j(x^k(t),t))\big) - \alpha_i(x^k(t)-\mu_i(0))^2 - \beta(u^k(t))^2, \end{aligned} \tag{2}$$

where the coefficient α_i is the stubbornness level of population $i \in I$, with $0 \le \alpha_i \le 1$. The cost functional c^k is made by three components; the first, weighted by $(1-\alpha_i)$, measures the *popularity* of opinion $x^k(t)$ according to the distribution of the other populations. The relevance of population j over opinion of player k belonging to population i is given by the coefficient ν_{ij}.[1]

[1] The choice of a logarithmic cost is commonly used when one wishes to describe a so-called *crowd-seeking* behavior on the part of the players (see, e.g., [14]). In this context,

Note that $\nu_{ij} = 0$ if populations i and j are not directly connected, so that agents of population i are not directly influenced by agents in population j. The second component of c^k, weighted by α_i, is related to the distance of $x^k(t)$ from the initial average opinion of the domestic population. As previously stated, this term accounts for coherence (or vicinity) between agents of the same population. The last term is a cost related to the action control $u^k(t)$, with $\beta > 0$ measuring the unitary cost.

In summary, heterogeneity among populations derives from different levels of stubbornness and different initial distribution of opinions. Moreover, because of the network structure, individuals choose their strategies considering the actions of their neighbors, namely, the individuals of the neighbor populations in a weighted way.

Eventually, each player maximizes the following infinite horizon discounted objective (expected utility) function

$$J(x_0^k) := \sup_{u^k \in \mathcal{U}} \mathbb{E}\{\int_0^\infty e^{-\rho t} c^k(x^k(t), u^k(t), m(t))dt\}, \tag{3}$$

where $\rho > 0$ measures impatience.

Depending on the value of the parameters, the model captures two radically distinct types of population: a *hard core* stubborn population or a *most gregarious* population. The players of the hard core stubborn population are characterized by $\alpha_i = 1$, then they tend to converge to their population's initial average opinion $\mu_i(0)$. Conversely, the players of a *most gregarious* pop-

the logarithmic function expresses the fact that the more players reach a consensus on state $x_i(\cdot)$, the smaller the marginal utility is of each new entrant player with same state $x_i(\cdot)$.

ulation are extremely susceptible to the inputs received from their neighbors and are characterized by $\alpha_i = 0$: their level of stubbornness is null. Populations characterized by $0 < \alpha_i < 1$ are called *partially stubborn*.

One has yet to determine the set of admissible controls and to fix suitable initial conditions of the populations' distributions. Unfortunately, to solve (3) in all generality would involve the solution of a system of coupled PDEs. This operation is usually impractical from a computational point of view. We refer to Appendix A for a general treatment of the problem and the derivation of the relative PDEs. Because of the complexity of the optimization problem, as noticed in [16], it is unlikely that the agents will follow such an approach. Having all of this in mind, it is convenient to restrict attention to a smaller class of controls, called affine control strategies. This class of strategies, besides being economically sound, is easy to deal with. We will show that we can determine the optimal control strategies within this class without resorting to any iterative computing process.

Given the above arguments we frame our model within the following assumption.

Assumption 1

1.a) Gaussian initial distributions*: At time 0, all the populations have a Gaussian distribution:*

$$m_i(x,0) = \frac{1}{\sigma_{0i}\sqrt{2\pi}} e^{-\frac{(x-\mu_{0i})^2}{2\sigma_{0i}^2}}, \quad \forall i \in I$$

where $\sigma_{0i} > 0$ and μ_{0i} are constant parameters for all $i \in I$.

1.b) Affine control strategies*: Agents adopt affine control strategies. Consider agent k belonging to population $i(k) = i$. The agent tracks a time varying weighted sum of $\mu_i(t)$, $\mu_j(t)$, for all j and $\mu_i(0) = \mu_{0i}$, namely, a strategy of type*

$$u_i(x^k(t), t) = d_i(t)\,[a_i(t)\mu_i(t) + \sum_{j\in I} b_{ij}(t)\mu_j(t) + c_i(t)\mu_{0i} - x^k(t)], \qquad (4)$$

where $d_i(t) > 0$ and $a_i(t), b_{ij}(t), c_i(t)$ are non negative for all t and

$$a_i(t) + \sum_{j\in I} b_{ij}(t) + c_i(t) = 1.$$

Some remarks on u_i are needed. First of all note that it encompasses the driving forces described above: the tendency to align with the most popular opinions (measured here in terms of the average populations' opinions) and the principle of preserving the initial average of the domestic population.[2] Moreover, it has the advantage of sharing the form of well-known *consensus protocols*. Thus, we can affirm that equilibrium strategies of type (4) drive the opinion of a single player towards a point in the convex hull of its own current opinion, its own population's initial mean opinion, and everybody else's opinion with respect to only the adjacent populations. In addition, we can benefit from convergence results used in the context of consensus dynamics [20]. Finally, note that u_i does not depend on the specific agent k but rather on the statistics of the population $i(k)$. This allows for the description of a system of N agents in terms of a system of n equations, where $n \ll N$.

[2] It is fairly accepted in the context of social interactions to assume that payoffs are linear in the average choice of the population (see, e.g., [19] for a reference contribution in the context of binary choice models).

Before computing the optimal control strategies, we describe a large class of controls that preserve the Gaussianity of the initial distributions. We refer to this class as *Gaussian Distribution Preserving Strategies* (GDPS).

Lemma 2.1 (Gaussian Distribution Preserving Strategies) *Suppose Assumption (1.a) holds. Then, a population i has a probability density function equal to*

$$m_i(x,t) = \frac{1}{\sigma_i(t)\sqrt{2\pi}} e^{-\frac{(x-\mu_i(t))^2}{2\sigma_i^2(t)}}, \tag{5}$$

for suitable differentiable functions $\mu_i(t)$ and $\sigma_i(t)$, provided that each player k of population i applies a control such as

$$u_i(x^k,t) = \hat{C}(t) e^{\frac{(x^k-\mu_i(t))^2}{2\sigma_i^2(t)}} + \left(\frac{2\sigma_t(t)\dot{\sigma}_i(t) - \xi_i^2}{2\sigma_i^2(t)}\right)(x^k - \mu_i(t)) + \dot{\mu}_i(t), \tag{6}$$

where $\hat{C}(t)$ is a suitable function of time.

Proof First of all, let us state and solve an *Inverse Fokker-Planck-Kolmogorov problem.* Consider $m_i(x,t) : S \times \mathbb{R}^+ \to \mathbb{R}$, of class $\mathcal{C}^{2,1}$, where $S \subseteq \mathbb{R}$ is an open set. Assume that m_i is a population density function, such that $m_i(x,t) \neq 0$ for all (x,t). Consider the unidimensional Fokker-Planck-Kolmogorov problem

$$\partial_t m_i - \frac{1}{2}\xi_i^2 \partial_{xx}^2 m_i = -\partial_x (u_i m_i). \tag{7}$$

The following control is the solution of (7)

$$u_i = \frac{1}{m_i}(C(t) + \frac{1}{2}\xi_i^2 \partial_x m_i - \int_{x_{0i}}^{x_i} \frac{\partial m_i}{\partial t} dx_i), \tag{8}$$

where $C(t)$ is an arbitrary function of time.

To prove this fact, note that equation (7) can be rewritten as the following first-order linear differential equation in u_i

$$\partial_x u_i = -\frac{1}{m_i}\partial_x m_i u_i - \frac{1}{m_i}(\partial_t m_i - \frac{1}{2}\xi_i^2 \partial_{xx}^2 m_i). \tag{9}$$

Given the assumptions on m_i, the coefficients $-\frac{1}{m_i}\partial_x m_i$ and $-\frac{1}{m_i}(\partial_t m_i - \frac{1}{2}\xi_i^2\partial_{xx}^2 m_i)$ are continuous functions on S. On the other hand, equation (8) can be interpreted as a set of ordinary differential equations with respect to the variable x and parameterized in t. Moreover, for each feasible t, the continuity of the equation coefficients guarantees the existence of a global solution. In addition, if for each feasible time t, an initial condition $u_{x_t t,i}$ for a point $(x_t, t) \in S$ is given, then such a solution is unique. Finally, observe that u_i, as defined in (8), solves (9); moreover, for each (x_t, t), we can fix $C(t)$ such that $u_i(x_t, t) = u_{x_t t,i}$.

We then apply the Inverse Fokker-Planck-Kolmogorov problem to the case of the Gaussian densities as defined in (5). The solution (8) of (7) in this case, turns out to be exactly (6). □

In the next lemma, we show that affine controls of Assumption 1.b, are GDPS. Thanks to this property, we are also able to characterize the time varying means and variances of the Gaussian populations' distributions.

Lemma 2.2 *Under Assumption 1, admissible controls are GDPS. Moreover, for all $t \geq 0$ and all $i \in I$,*

$$m_i(x,t) = \frac{1}{\sigma_i(t)\sqrt{2\pi}} e^{-\frac{(x-\mu_i(t))^2}{2\sigma_i^2(t)}}, \tag{10}$$

where $\mu_i(t)$ and $\sigma_i(t)$ are, respectively, the unique solutions to

$$\dot{\sigma}_i^2(t) = -2d_i(t)\sigma_i^2(t) + \xi_i^2, \quad \sigma_i^2(0) = \sigma_{0i}^2 \tag{11a}$$

$$\dot{\mu}_i(t) = d_i(t)\big[\sum\nolimits_{j\in I} b_{ij}(t)\mu_j(t) + c_i(t)\mu_{0i} - (\sum\nolimits_{j\in I} b_{ij}(t) + c_i(t))\mu_i(t)\big], \quad \mu_i(0) = \mu_{0i} \tag{11b}$$

and where functions b_{ij}, c_i and d_i have been defined in Assumption 1.

Proof As a consequence of Lemma 2.1, affine control strategies

$$u_i(x,t) = \hat{p}_i(t)x + \hat{q}_i(t), \tag{12}$$

where $\hat{p}_i$ and $\hat{q}_i$ are $\mathcal{C}^1$ functions of time, are GDPS. Indeed, (12) is obtained from (6), when $\hat{C}(t) = 0$, for all $t \geq 0$. Controls as in (4) can be easily rewritten in the form as in (12), where $\hat{p}_i(t) = -d_i(t)$ and

$$\hat{q}_i(t) = d_i(t)\big[\sum\nolimits_{j\in I} b_{ij}(t)\mu_j(t) + c_i(t)\mu_{0i} - \big(\sum\nolimits_{j\in I} b_{ij}(t) + c_i(t)\big)\mu_i(t)\big].$$

Being GDPS, they preserve Gaussianity of the initial distributions. Moreover, through a direct application of the Itô Lemma, it can be easily verified that in the case of affine control strategies, the solution of (1) is

$$x^k(t) = e^{\hat{P}_i(t)}\Big(x_0^k + \int_0^t e^{-\hat{P}_i(\tau)} q_i(\tau)d\tau + \xi_i \int_0^t e^{-\hat{P}(\tau)} dW^k(\tau)\Big),$$

with $i = i(k)$ and $\hat{P}_i(t) = \int_0^t \hat{p}_i(\tau)d\tau$. In addition, under the same hypothesis, the mean $\mu_i(t)$ and the variance $\sigma_i^2(t)$ of the population distribution evolve as follows:

$$\sigma_i^2(t) = e^{2\hat{P}_i(t)}\Big(\sigma_{0i}^2 + \xi_i^2 \int_0^t e^{-2\hat{P}_i(\tau)} d\tau\Big), \tag{13a}$$

$$\mu_i(t) = e^{\hat{P}_i(t)}\Big(\mu_{0i} + \int_0^t e^{-\hat{P}_i(\tau)} q_i(\tau)d\tau\Big). \tag{13b}$$

By differentiating equations (13), we obtain (11). □

Unfortunately, to our knowledge, it is not possible to explicitly determine the optimal control among the class of affine control strategies, i.e., the optimal time varying coefficients a_i, b_{ij}, c_i and d_i characterizing it. Nevertheless, we will see in the next section that this can be done by assuming that agents are *myopic*, in the sense of [16]. By myopic we mean that, when choosing their strategy at time t, agents consider the future populations' distributions as *constant* and equal to the present ones.

3 The Case of Myopic Agents

Within the class of linear control strategies, we determine analytically those that are optimal if the agents adopt the myopic perspective discussed in [16]. We call such strategies *myopic equilibrium controls*. These strategies are based on the assumption that agents, when taking their decisions at time t, consider the future populations' distributions as *constant* or, put differently, that for all $s \geq t$ the populations' distributions have reached their invariant state. According to this latter interpretation, myopic equilibrium controls can be seen as "long run solutions" of (3) in the sense that they involve a set $\{(m_i, u_i) : i \in I\}$ of functions that satisfy the following properties. For all $i \in I$, $m_i(\cdot, t)$ and $u_i(\cdot, t)$ converge to a stationary distribution $\mu_i^{eq}(\cdot)$ and a stationary control function $u_i^{eq}(\cdot)$, such that the set $\{(m_i^*(\cdot, t) = \mu_i^{eq}(\cdot), u_i^*(\cdot, t) = u_i^{eq}(\cdot)); \forall t \geq 0, i \in I\}$ is solution of (3) given the initial conditions $m_{0i}(\cdot) = \mu_i^{eq}(\cdot)$, for all $i \in I$.

Assumption 2 *Players optimize myopically, that is, at each time $t > 0$, players decide their control policy assuming that the populations' distribution will remain constant in time. In other words, $m_j(\cdot, s) = m_j(\cdot, t)$, for all $j \in I$ and $s > t$.*

The above assumption implies that mean and variance of each population in I remain constant. Under Assumption 2, it is not difficult to see that each player k may determine its state-feedback control strategy by solving the following *myopic optimal control problem* for each time t.

$$J_i(x^k(t), t) =$$
$$\sup_{u_i \in \mathcal{U}} \mathbb{E}\Big\{ -\int_t^\infty e^{-\rho\tau}\Big[(1-\alpha_i)\sum_{j\in I}\frac{\nu_{ij}}{2}\frac{(x^k(\tau)-\mu_j(t))^2}{\sigma_j^2(t)} + \alpha_i(x^k(\tau)-\mu_{0i})^2 + \beta(u_i(\tau))^2\Big]d\tau\Big\} \tag{14}$$

where $x^k(t)$ is given, $i = i(k)$, and

$$dx^k(\tau) = u_i(x^k(\tau), \tau)d\tau + \xi_i\, dW^k(\tau), \quad \forall \tau \geq t.$$

Note that, in (14), we can disregard the term $\sum_{j\in I} -\frac{\nu_{ij}}{2}\ln(2\pi\sigma_j^2)$ of the running cost (2) because it is constant.

The following theorem identifies the state-feedback optimal control by specifying the functions a_i, b_{ij}, c_i and d_i for the affine strategies as in (4).

Theorem 3.1 *Under Assumption 1 and 2, there exist optimal $(u_i^*(\cdot, t), m_i^*(\cdot, t))_{t\geq 0}$, $i \in I$ such that*

$$u_i^*(x^k(t), t) = d_i(t)\big(a_i(t)\mu_i(t) + \sum_{j\in I} b_{ij}(t)\mu_j(t) + c_i(t)\mu_{0i} - x^k(t)\big), \tag{15}$$

where

$$d_i(t) = \sqrt{\frac{\rho^2}{4} + \frac{\gamma_i(t)}{\beta}} - \frac{\rho}{2}, \tag{16a}$$

$$a_i(t) = \frac{(1-\alpha_i)\frac{\nu_{ii}}{2\sigma_i^2(t)}}{\gamma_i(t)}, \tag{16b}$$

$$b_{ij}(t) = \frac{(1-\alpha_i)\frac{\nu_{ij}}{2\sigma_j^2(t)}}{\gamma_i(t)}, \tag{16c}$$

$$c_i(t) = \frac{\alpha_i}{\gamma_i(t)}, \tag{16d}$$

being

$$\gamma_i(t) = \left[(1-\alpha_i)\left(\frac{\nu_{ii}}{2\sigma_i^2(t)} + \sum_{j\in I}\frac{\nu_{ij}}{2\sigma_j^2(t)}\right) + \alpha_i\right].$$

Moreover, the associated populations' distributions are Gaussian with means and variances evolving according to equation (11)*.*

Proof The control policy (15) is an immediate consequence of the fact that the players determine their control policy solving Problem (14) under Assumption 2 at each time $t \geq 0$. Problem (14) is a standard linear quadratic problem yielding the optimal control

$$\begin{aligned} u_i(x^k(\tau),\tau) &= -K\left(x^k(\tau) - \frac{(1-\alpha_i)\left(\frac{\nu_{ii}\mu_i(t)}{2\sigma_i^2(t)} + \sum_{j\in I}\frac{\nu_{ij}\mu_j(t)}{2\sigma_j^2(t)}\right) + \alpha_i\mu_{0i}}{(1-\alpha_i)\left(\frac{\nu_{ii}}{2\sigma_i^2(t)} + \sum_{j\in I}\frac{\nu_{ij}}{2\sigma_j^2(t)}\right) + \alpha_i}\right) \\ &= -K\left(x^k(\tau) - \frac{(1-\alpha_i)\left(\frac{\nu_{ii}\mu_i(t)}{2\sigma_i^2(t)} + \sum_{j\in I}\frac{\nu_{ij}\mu_j(t)}{2\sigma_j^2(t)}\right) + \alpha_i\mu_{0i}}{\gamma_i(t)}\right). \end{aligned} \tag{17}$$

In the above, $K = \frac{1}{\beta}P$, where P is the solution of the following unidimensional Riccati equation:

$$\frac{1}{\beta}P^2 + \rho P - \gamma_i(t) = 0,$$

which has solution $P = \beta d_i(t)$, where $d_i(t)$ is given in (16a). The values of $a_i(t)$, $b_{ij}(t)$, and $c_i(t)$ in (16) are consequences of the fact that (17) presents an affine structure as in (4) in Assumption 1. The Gaussianity of the populations' distributions and the time evolution of means and variances as in equation (11), follow from Lemma 2.2, thanks to the fact that controls are affine. □

It follows from the above theorem, that a hard core stubborn population i is characterized by $a_i(t) = b_{ij}(t) = 0$ and $c_i(t) = 1$, for all $t \geq 0$ and $j \in I$, whereas a most gregarious population i is characterized by $c_i(t) = 0$, for all $t \geq 0$. It is worth noting that $d_i(t)$ is a function of only the variance of the population i and of its neighboring ones. This last observation will become useful when studying the asymptotic behaviors of the populations in I.

Finally, notice that the set $(u_i^*(\cdot, t), m_i^*(\cdot, t))_{t\geq 0}$, solution to the optimization problem (14), is a *Nash-Mean Field Equilibrium* as described in Appendix A. This means that, when playing optimal strategies, any agent of population i does not benefit from changing its control policy if the control policies, and therefore also the distributions, of the other populations are fixed to respectively u_j^* and m_j^* for all $j \in I \setminus \{i\}$.

3.1 Elementary Macroscopic Behavioral Patterns

In the previous section we have derived myopic equilibrium strategies as solution of the optimal control problem (14). These equilibrium strategies govern the *microscopic* evolution of the players' opinions given in (1). We now provide details on the macroscopic dynamics resulting from the above microscopic behaviors. In particular, we detect three different *elementary behavioral patterns*, all of which are connected to classical consensus dynamics.

The next corollary establishes that disagreement or polarization of opinions, arise even in response to the presence of just two partially stubborn populations.

Corollary 3.1 (Equilibrium of Means and Disagreement) *Consider two populations i and j in I which are at least partially stubborn. Assume, moreover, that $\mu_{0i} \neq \mu_{0j}$. Then, the means of the two populations cannot converge to the same value.*

Proof By virtue of Theorem 3.1, we know that populations' distributions evolve according to (11b). Assume by contradiction that the means of the two populations converge to the same value: $\mu_i^{eq} = \mu_j^{eq} = \mu^{eq}$. If we are at the equilibrium, (11b) implies

$$d_i(\sum_{k \in I} b_{ik}\mu^{eq} + c_i\mu_{0i} - (\sum_{k \in I} b_{ik} + c_i)\mu^{eq}) = 0 \Leftrightarrow d_i c_i(\mu_{0i} - \mu^{eq}) = 0;$$
$$d_j(\sum_{k \in I} b_{jk}\mu^{eq} + c_j\mu_{0j} - (\sum_{k \in I} b_{jk} + c_j)\mu^{eq}) = 0 \Leftrightarrow d_j c_j(\mu_{0j} - \mu^{eq}) = 0.$$

The above conditions hold iff $\mu_{0i} = \mu_{0j}$, in contradiction with the lemma's hypothesis. □

In the next theorem we show how to take advantage of the relationship between strategies (4) and consensus algorithms as in [20]. To this aim, consider an extended graph $G' = (I \cup \{01, \ldots, 0n\}, E \cup \{(1, 01), \ldots, (n, 0n)\},)$ as illustrated in Figure 1. Graph G' enriches G with a new node per each population j, representing the initial average value μ_{0j} (we call it $0j$); it also shows a new directed edge $(j, 0j)$. Nodes and edges of the original graph G are in blue, while the new nodes and edges of G' are in red. The construction of an extended

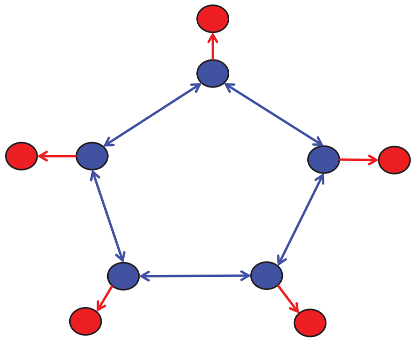

Fig. 1 Extended communication graph G' obtained from G by adding the red nodes and the directed red edges.

graph will contribute a new perspective on the equilibrium strategies, these being now rewritten as *consensus protocols* [20]. With the above graph in mind, let us denote by $\Theta(t) = [\mu_1(t), \mu_{01}(t), \ldots, \mu_n(t), \mu_{0n}(t)]^T$ where $\mu_{0n}(t)$ is fictitiously made dependent on time. Let us also introduce

$\Theta_0 = [\mu_{01}, \mu_{01}, \ldots, \mu_{0n}, \mu_{0n}]^T$. Then dynamics (11b) can be written in vector form as

$$\dot{\Theta}(t) = -L_{G'}\,\Theta(t), \quad \Theta(0) = \Theta_0$$

where $L_{G'}$ denotes the Laplacian matrix of the graph G'. The dynamics have the form of classical consensus dynamics. Once we realize this, we can extend to general opinion dynamics as in (11b), taking advantage of results pertaining to the consensus literature [20].

We show the powerfulness of these techniques by deriving some representative and elementary behavioral patterns on the social network. More specifically, we now describe three behavioral patterns, which pertain to different scenarios, involving i) one or more hard core populations, ii) populations which are all most gregarious, and iii) one hard core population and the rest of the populations most gregarious. A sufficient condition to have convergence of mean values of the populations' distributions, is that the populations' initial variances are at the equilibrium at time 0, or, put differently, that we start monitoring the process once the variances have reached suitable steady states.

Theorem 3.2 (Elementary Behavioral Patterns) *Suppose*

$$\sigma_{0i}^2 = \frac{\xi_i^2}{2d_i(0)}, \quad \forall i \in I. \tag{18}$$

The following facts hold true.

1. *(*hard core stubborn population*) If a population $i \in I$ is hard core stubborn ($\alpha_i = 1$) then*

$$\mu_i(t) = \mu_{0i}, \quad t \geq 0. \tag{19}$$

2. *(*most gregarious populations*) If all the populations $i \in I$ are most gregarious, the means of the populations converge exponentially to a single consensus value, namely,*

$$\mu_i^{eq} = \mu^{eq}, \quad \forall i \in I. \tag{20}$$

Furthermore, if graph G is balanced (or undirected)*, expectations converge exponentially fast to the average consensus value, i.e.,*

$$\mu^{eq} = \frac{1}{n} \sum_{k=1}^{n} \mu_{0k}, \tag{21}$$

with known bound for the convergence rate.

3. *(*single partially stubborn population*) If there exists a population $j \in I$ that is not most gregarious ($\alpha_j > 0$) and all the remaining populations $i \in I \setminus \{j\}$ are most gregarious ($\alpha_i = 0$), then for all $i \in I \setminus \{j\}$*

$$\mu_i^{eq} = \mu_j^{eq} = \mu_{0j}. \tag{22}$$

However, if population j is not hard core stubborn and b_{ji} is not equal to 0 for all $i \in I \setminus \{j\}$, in general, there could exist some $t > 0$, such that $\mu_j(t) \neq \mu_{0j}$.

Proof Let us start by noting that according to (13a) and (11a), condition (18) is exactly what is needed to ensure that the populations' variances are in equilibrium at time 0. In this case, the values of a_i, b_{ij}, c_i in (16) are constant in time, for all $i \in I$. Therefore, also the populations' means $\mu_i(t)$, solutions of (11b), converge to (possibly different) finite values, i.e.,

$$\lim_{t\to\infty} \mu_i(t) = \mu_i^{eq}. \tag{23}$$

We can interpret conditions (11b) as the equations of a stable dynamic consensus system (see [20]), whose state is $[\mu(t), \mu_0(t)]$, where $\mu(t) = \{\mu_i(t), i \in I\}$ and $\mu_0(t) = \{\mu_{0i}, i \in I\}$.

1. If $\alpha_i = 1$, then, (11b) becomes $\dot{\mu}_i(t) = d_i c_i(\mu_{0i} - \mu_i(t))$, whose only solution is $\mu_i(t) = \mu_{0i}$ for all $t \geq 0$.
2. If $c_i = 0$, for all $i \in I$, then equations (11b) can be rewritten as

$$\begin{bmatrix} \dot{\mu}_1(t) \\ \vdots \\ \dot{\mu}_n(t) \end{bmatrix} = -L_G \begin{bmatrix} \mu_1(t) \\ \vdots \\ \mu_n(t) \end{bmatrix} \text{ where } (L_G)_{ij} = \begin{cases} d_i b_{ij} & \text{if } j \neq i, \\ -d_i \sum_{j\in I} b_{ij} & \text{if } j = i. \end{cases} \tag{24}$$

Equation (24) describes an autonomous dynamic system whose state components are the means of the populations. Since we assume that G is strongly connected, the Laplacian matrix L_G of G is a transition rate matrix with a right eigenvector equal to $\mathbf{1} := [1, 1, \ldots, 1]^T$ and a corresponding null eigenvalue. In other words, we have $L_G \mathbf{1} = 0$. Consequently, the components of its state, that is the means of all the populations, converge to a single consensus value [20].

To prove the second part of the statement, let $\theta(t) = [\mu_1(t), \ldots, \mu_n(t)]^T$ and $\theta_0 = [\mu_{01}, \ldots, \mu_{0n}]^T$. In other words, (24) reads

$$\dot{\theta}(t) = -L_G\, \theta(t), \quad \theta(0) = \theta_0.$$

Now, let us consider the *disagreement function* (see [20])

$$\phi(\theta) = \frac{1}{2} \theta^T L_G \theta.$$

It is well-known that, if the graph G is balanced (or undirected), then $\mathbf{1}$ is a left eigenvector for L_G; therefore, the average $\frac{1}{n} \sum_{k \in I} \mu_k(t)$ is invariant and this proves the convergence towards the average consensus. In addition, dynamics (24) are equivalent to the *gradient descent law* (see [20])

$$\dot{\theta}(t) = \partial_\theta \phi(\theta(t)).$$

Let λ_2 be the second smallest eigenvalue of L_G, also known as Fiedler eigenvalue, and let the disagreement vector be $\delta = \theta - \mathbf{1}(\frac{1}{n} \sum_{k \in I} \mu_{0k})$. Exponential stability derives from using the Lyapunov function $\Phi(t) = \delta^T \delta$

and noting that it holds that

$$\dot{\Phi}(t) = -2\delta^T L\delta \leq -2\lambda_2\delta^T\delta = -2\lambda_2\Phi.$$

The latter equation implies that the convergence rate is bounded from below by the Fiedler eigenvalue.

3. First, we note that the graph $G'' = (I \cup \{j0\}, E \cup \{(j, j0)\})$ has a directed spanning tree as G is strongly connected. Here, node $j0$ acts as leader. Then, even in this case, equations (11b) describe an autonomous dynamic system in which the means of all the populations converge to a single consensus value. Then, as regards population j, the only stable condition is that $\mu_j(t) = \mu_{0j}$.

   To prove the second part of point 3, note that if $\sum_{i\in I} b_{ij} \neq 0$ and if $\frac{\sum_{i\in I} b_{ij}\mu_i(t)}{\sum_{i\in I} b_{ij}} \neq \mu_{0j}$ for some $t > 0$, then $\dot{\mu}_j(t) \neq 0$, as the control implemented by the players of population j is not symmetric with respect to μ_{0j}. Thus, the mean of the population swings towards $\frac{\sum_{i\in I} b_{ij}\mu_i(t)}{\sum_{i\in I} b_{ij}}$.

□

Let us note that case 1. of Theorem 3.2 does not require any assumption of strong connectedness of graph G. In fact, result (19) trivially holds also in case a population $i \in I$, independently of the value of α_i, is characterized by $\nu_{ij} = 0$ for all $j \in I_i$, that is, when population i is not strongly connected to any other population.

Figure 2 depicts a simple example of graph G' for each case treated in the previous results. Indeed, G (the blue graph) is strongly connected in all the situations except for the one in panel b). In panel a), we have the case dealt with in Corollary 3.1, characterized by two partially stubborn populations. In panel b), we see an example of case 1. of Theorem 3.2, where both a hard core stubborn population and a population not strongly connected to any other population are present. Panel c) represents case 2. of Theorem 3.2, where all populations are most gregarious and no directed link connects them to the red nodes representing their initial mean values. Finally, in d), we identify case 3. of Theorem 3.2, where we have just one partially stubborn population and the others are all most gregarious.

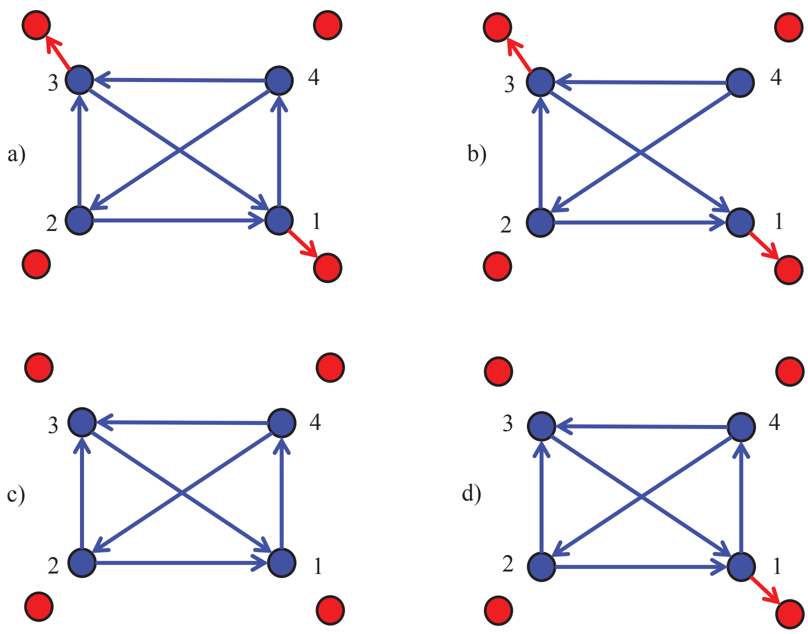

Fig. 2 (a) two partially stubborn populations (populations 1 and 3); (b) one hard core stubborn population (population 3) and one partially stubborn population (population 1) not strongly connected to any other population; (c) most gregarious populations (populations 1, 2, 3 and 4); (d) a single partially stubborn population (population 1)

In the parlance of opinion dynamics, both cases 2. and 3. of Theorem 3.2 are referred to *consensus* of opinions. This depends on the fact that we have assumed graph G to be strongly connected and there is no more than one at least partially stubborn player. If we relax the strong connectivity assumption in favor of multiple strongly connected components, we can have *polarization* (only two strongly connected components) or even *plurality* of opinions (several strongly connected components). The results provided above can also be extended to time-varying graphs $G(t) = (I, E(t))$, where now the edge set $E(t)$ depends on time. Sufficient conditions involve then the existence of a spanning tree for the union graph over a period, namely the graph with the edge set obtained by the union of the infinite edge sets occurring in a given interval (see [20]). Of particular interest, when dealing with time-varying topologies, is the case of periodical changes of the spanning tree. In this setting, one can model different populations acting as leaders in different periods. This situation can be obtained as a combination of the elementary behavioral patterns as described in Theorem 3.2. We skip the technical details and refer the reader to Section 4.2, where some numerical instances will be run to graphically illustrate the behavior of the system in the presence of time-varying leaders.

As explained, Theorem 3.2 rests on the assumption that the distributions' variances reach (or start by) an equilibrium. In the next section, we discuss the feasibility of the sufficient conditions given in equation (18). In particular, we show that under mild assumptions it can actually be verified.

3.2 On the Feasibility of the Populations' Variance Equilibrium

We now explore some conditions for the existence of a value $d_{0i} > 0$ such that, whenever $d_i(0) = d_{0i}$, both (16a) and (18) hold at time $t = 0$.

We initially consider populations characterized by the same level of stubbornness and subject to the same disturbance, in which case we denote the populations as *uniform*. This corresponds to set $\alpha_i = \alpha$ and $\xi_i = \xi$ for all $i \in I$. Under these hypotheses we have that $\sigma_i^2(0) = \frac{\xi^2}{2d_0}$, hence we can rewrite (16a) at time 0 as

$$\beta\rho^2 + 4\left((1-\alpha)\frac{d_0}{\xi^2}\sum_{j\in I}\nu_{ij} + \alpha\right) = 4d_0^2\beta + 4d_0\beta\rho + \beta\rho^2 \quad \Leftrightarrow$$
$$\Leftrightarrow \quad (1-\alpha)\frac{d_0}{\xi^2} + \alpha = d_0^2\beta + d_0\beta\rho \quad \Leftrightarrow \quad d_0^2\beta - d_0\Big(\frac{1-\alpha}{\xi^2} - \beta\rho\Big) - \alpha = 0.$$

The latter equation is a second order polynomial which admits (real) solutions in each of the following three situations.

- Populations are most gregarious ($\alpha = 0$). In this case, we have the solution
$$d_0 = \frac{1}{\beta\xi^2} - \rho,$$
which is feasible, i.e. strictly greater than zero, only for a sufficiently small ρ. This result is coherent with related results in [21], and we refer the interested reader to that paper for more details on the general case of most gregarious populations.
- Populations are hard core stubborn ($\alpha = 1$). In this case, the solution is given by
$$d_0 = \sqrt{\frac{\rho^2}{4} + \frac{1}{\beta}} - \frac{\rho}{2} > 0, \quad \forall\rho > 0.$$

Note that this result also extends to the case of non-uniform populations, namely, populations that are characterized by different levels of stubbornness and are subject to different disturbances.

– Populations are partially stubborn. In this case, we have the general solution

$$d_0 = \sqrt{\left(\frac{\rho}{2} - \frac{1-\alpha}{2\beta\xi^2}\right)^2 + \frac{\alpha}{\beta}} - \left(\frac{\rho}{2} - \frac{1-\alpha}{2\beta\xi^2}\right) > 0, \quad \forall \rho > 0$$

It appears that a minimum level of stubbornness is sufficient to guarantee the existence of an equilibrium for the populations' variances.

Existence of an equilibrium for the populations' variances can be proved even in the case of just two non-uniform populations, when one of them is at least partially stubborn. In this case, we have to prove the existence of positive values d_{01} and d_{02} that solve the following equations

$$d_{0i}^2\beta - d_{0i}\left(\frac{(1-\alpha_i)\nu_{ii}}{\xi_i^2} - \beta\rho\right) - \left(\alpha_i + d_{0j}\frac{(1-\alpha_i)\nu_{ij}}{\xi_j^2}\right) = 0, \quad i, j = 1, 2,\ i \neq j.$$

Then, assuming that the first population is at least partially stubborn ($\alpha_1 > 0$), we can introduce two real and continuous functions g and h defined as

$$g(x) = \sqrt{\left(\frac{\rho}{2} - \frac{(1-\alpha_1)\nu_{11}}{\beta\xi_1^2}\right)^2 + \left(\frac{\alpha_1}{\beta} + x\frac{(1-\alpha_1)\nu_{12}}{\beta\xi_2^2}\right)} - \left(\frac{\rho}{2} - \frac{(1-\alpha_1)\nu_{11}}{\beta\xi_1^2}\right),$$

$$h(x) = x^2\beta - x\left(\frac{(1-\alpha_2)\nu_{22}}{\xi_2^2} - \beta\rho\right) - \left(\alpha_2 + g(x)\frac{(1-\alpha_2)\nu_{21}}{\xi_1^2}\right).$$

Note that, $g(d_{02}) = d_{01}$. It is not difficult to see that an equilibrium for the populations' variances exists if the following conditions hold: 1) there exists $d_{02} > 0$ for which $h(d_{02}) = 0$; 2) $d_{01} = g(d_{02}) > 0$. To show this, consider that, for $d_{02} \geq 0$, $g(d_{02})$ is positive. Then, since $h(0) < 0$ and $\lim_{d_{02}\to+\infty} h(d_{02}) =$

$+\infty$, by continuity there must exists $d_{02} > 0$ for which condition 1) holds. In addition, the positivity of $g()$ for $d_{02} > 0$ guarantees that condition 2) holds too. More elaborate but similar arguments can be used to generalize the above results to the case of more than two populations.

4 Hard Core vs. Most Gregarious Populations

In this section we concentrate our attention on a specific situation where a most gregarious population deals with a hard core one. Such a scenario mirrors games where the latter is, for instance, a leader nation and the former a minor nation of followers. Indeed, we provide a detailed stochastic analysis of the microscopic behavior of the players of the most gregarious population, studying the relationships between the behavior of each single player and that of the population as a whole. In particular, we construct error dynamics which account for the deviation of the opinion of the single player from the average value of the population. We interpret such error dynamics as stochastic processes and provide some bounds on their first and second moments.

For the sake of simplicity, we assume that the hard core population j (characterized by $\alpha_j = 1$) has already reached an equilibrium distribution characterized by zero mean $\mu_{0j} = 0$ and unitary variance $\sigma_{0j} = 1$. In addition, we assume that also the variance of the most gregarious population i (characterized by $\alpha_i = 0$) has already reached its equilibrium value $\sigma_{0i} > 0$. Under these assumptions, thanks to (15), the optimal control for agent k belonging to the most gregarious population simplifies to

$$u_i^*(x^k(t),t) = \phi_a(t)(\mu_i(t) - x^k(t)) - \phi_b(t)x^k(t), \tag{25}$$

where $\phi_a(t) = d_i(t)\, a_i(t)$ and $\phi_b(t) = d_i(t)\, b_{ij}(t)$.

In the next sections we put emphasis on the implications of the stochasticity present in the finite dimensional model with N agents. In particular, in Section 4.1 we compute bounds on the speed of convergence of the most gregarious population average behavior to zero, characterizing also deviations of single opinions from average values. In Section 4.2 we run simulations of the finite dimensional system to show how the speed of convergence of means and variances depend on the parameters of the model. Since we are interested in the behavior of the most gregarious population, from now on, we drop indexes i and j.

4.1 The Finite Dimensional Model

In what follows, we denote by $Y(t) = [Y^1(t), \ldots, Y^N(t)]^T$ the state vector of the N players indexed by $k = 1, \ldots, N$. It describes the time evolution of the players' states in accordance with the Stochastic Differential Equation (SDE)

$$dY(t) = [\phi_a(t)\,(\mu_N(t)\mathbf{1} - Y(t)) - \phi_b(t)\, Y(t)]\, dt + \xi\, dW(t), \tag{26}$$

where we make explicit the dependence on N of the average opinion $\mu_N(t)$.

In particular, we may find useful to rewrite (26) by making use of a stochastic matrix. To this aim, let $\Pi = -\phi_a L + I$, where L is the Laplacian of a strongly completely connected graph, the latter modeling the interconnection between the players of the most gregarious population. Note that

$$\Pi = \Pi^T, \quad \Pi \mathbf{1} = \mathbf{1}. \tag{27}$$

Then, we can rewrite (26) as

$$dY(t) = [(\Pi - I)\, Y(t) - \phi_b(t)\, Y(t)]\, dt + \xi dW(t). \tag{28}$$

With the above premise, the SDE is linear and time-varying which allows us to cast our analysis within the framework of stochastic stability (see [22]). Let us recall the definition of stability in pth moment.

Definition 4.1 (Stability in pth moment) (cf. Definition (11.3.1) in [23]) The solution $\psi(t)$ of a stochastic differential equation is said to be *stable in pth moment*, $p > 0$, iff given $t_0 > 0$ and $\varepsilon > 0$, there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|x(0)\| \leq \delta$ guarantees that

$$\mathbb{E}\left\{\sup_{t \geq t_0} \|\psi(t)\|^p < \varepsilon\right\}.$$

Setting $p = 1$ or 2 we have *stability in mean* or *in mean square*, respectively. Furthermore, it is well-known that mean square stability implies mean stability ([23], p. 188).

The next theorem proves mean square stability for the average opinion.

Proposition 4.1 *The average opinion is mean square stable. In particular, there exists $t_0 > 0$, such that*

$$\mathbb{E}\left\{\sup_{t \geq t_0} \|\mu_N(t)\|^2 < \varepsilon\right\}, \quad \textit{where } \varepsilon = \sqrt{\frac{1}{2\phi_b(t)} \frac{\xi^2}{N^2}}. \tag{29}$$

Proof The basic ideas of this proof are borrowed from Lyapunov stochastic stability theory (cf. [22], system (64.51)). Let us start by considering the dynamics for the average opinion, which is given by

$$\begin{aligned} d\mu_N(t) &= \frac{1}{N}\mathbf{1}^T dY(t) = \frac{1}{N}\mathbf{1}^T \left[(\Pi - I)Y(t) - \frac{q_2}{c_2}Y(t)\right] dt + \frac{1}{N}\mathbf{1}^T \xi dW(t) \\ &= -\frac{1}{N}\mathbf{1}^T \frac{q_2}{c_2} Y(t)dt + \frac{1}{N}\mathbf{1}^T \xi dW(t) = -\phi_b(t)\, \mu_N(t)dt + \frac{1}{N}\mathbf{1}^T \xi dW(t). \end{aligned} \tag{30}$$

This is a linear SDE and the properties of the corresponding stochastic process have been analyzed in [22]. Actually, the solution $\mu_N(t)$ to (30) is associated to the following infinitesimal generator (see [23])

$$\mathcal{L} = \frac{1}{2}\xi^2 \frac{d^2}{dm} - \phi_b\, m \frac{d}{dm}. \tag{31}$$

Let us take as candidate Lyapunov function $V(m) = \frac{1}{2}m^2$. The idea is to show that there exists a finite scalar κ and a level set $\mathcal{N}_\kappa = \{m \in \mathbb{R} |\, V(m) \leq \kappa\}$, such that the infinitesimal generator

$$\mathcal{L}V(\mu_N(t)) := \lim_{dt \to 0} \tfrac{\mathbb{E}V(\mu_N(t+dt)) - V(\mu_N(t))}{dt} < 0, \tag{32}$$

is negative for all $\mu_N(t) \notin \mathcal{N}_\kappa$. In this case, $V(\mu_N(t))$ is a supermartingale whenever $\mu_N(t)$ is not in $\mathcal{N}_\kappa$ and therefore, by the martingale convergence theorem, there exists $t > 0$ such that $V(\mu_N(t)) \leq \kappa$.

To verify (32), observe that from (30) we have

$$\mathcal{L}V(m) = -\phi_b\, m^2 + \frac{1}{2}\frac{\xi^2}{N^2}.$$

Therefore, there exists a $\hat{\kappa}$ big enough and finite that for every $\mu_N(t) \notin \mathcal{N}_{\hat{\kappa}}$, i.e., $\frac{1}{2}\mu_N(t)^2 > \hat{\kappa}$, we have $\phi_b \mu_N^2 > \frac{1}{2}\frac{\xi^2}{N^2}$. The latter implies $\mathcal{L}V(\mu_N(t)) < 0$

for all $\mu_N(t) \notin \mathcal{N}_{\hat{\kappa}}$, which proves that every level set $\mathcal{N}_\kappa$ where $\kappa \geq \hat{\kappa}$ is contractive.

The same reasoning proves that every level set $\mathcal{N}_\kappa$ where $\kappa \geq \hat{\kappa}$ is contractive. Thus, for every $\kappa \geq \hat{\kappa}$ there exists an $\varepsilon = \sqrt{2\kappa}$ for which the set $\{m \in \mathbb{R} | \, |m| \leq \varepsilon\}$ is contractive. A value for $\hat{k}$ is

$$\hat{k} := \min k \quad \text{s.t.} \quad \{m| \, V(m) = k\} \subset \{m|\phi_b \, m^2 > \tfrac{1}{2} \tfrac{\xi^2}{N^2}\}, \tag{33}$$

which returns

$$\hat{k} = \frac{1}{4\phi_b} \frac{\xi^2}{N^2}.$$

To prove (29), let us substitute $\hat{k} = \frac{1}{4\phi_b} \frac{\xi^2}{N^2}$ into $\varepsilon = \sqrt{2\kappa}$, obtaining $\varepsilon = \sqrt{\frac{1}{2\phi_b} \frac{\xi^2}{N^2}}$. □

A direct consequence of the above result is that $\mathcal{N}_{\hat{\kappa}}$ shrinks for increasing number of players and collapses asymptotically to the origin for N tending to infinity.

Corollary 4.1 *For $N \to \infty$ the mean opinion $\mu_N(t)$ asymptotically converges to zero.*

Our aim is now to analyze the convergence of the players' opinions to the average value. To this purpose, define the averaging matrix $\mathcal{M} = \frac{1}{N}\mathbf{1} \otimes \mathbf{1}$. Then for a given vector $Y(t)$ we have $\mathcal{M}Y(t) = (\frac{1}{N}\mathbf{1} \otimes \mathbf{1})Y(t) = \frac{1}{N}\mathbf{1}\mathbf{1}^T Y(t) = \mu_N(t)\mathbf{1}$. In other words $\mathcal{M}Y(t)$ is the vector all of those whose components are the average of the entries of $Y(t)$. The averaging matrix is useful to introduce the error vector $e(t)$ describing the deviations of the components of $Y(t)$ from their average. For the error vector we can write the expression below, which relates $e(t)$ to $Y(t)$:

$$e(t) = Y(t) - \frac{1}{N}\mathbf{1} \otimes \mathbf{1}^T Y(t) = Y(t) - \mu_N(t)\mathbf{1} = (I - \mathcal{M})Y(t).$$

The next result establishes that the error vector is bounded in probability.

Proposition 4.2 *For each $\pi > 0$ there exists an $\varepsilon(\pi) > 0$ such that*

$$\mathbb{P}(\|e(t)\|_\infty \leq \varepsilon(\pi)) > 1 - \pi. \tag{34}$$

Proof The time evolution of the error vector is represented by the SDE

$$\begin{aligned} de(t) &= (I - \mathcal{M})\left[(\Pi - I)Y(t) - \phi_b(t)\,Y(t)\right]dt + (I - \mathcal{M})\xi dW(t) \\ &= (\Pi - \mathcal{M})(I - \mathcal{M})Y(t)dt - e(t)dt - (I - \mathcal{M})\phi_b(t)\,Y(t)dt + (I - \mathcal{M})\xi dW(t) \\ &= \underbrace{(\Pi - \mathcal{M} - I - \phi_b(t)\,I)}_{A}\,e(t)dt + (I - \mathcal{M})\xi dW(t). \end{aligned}$$

The above SDE is linear and the corresponding stochastic process can be studied again in the framework of stochastic stability theory [22]. As before, we can consider the associated generator

$$\mathcal{L} = \frac{1}{2}\xi^2(I - \mathcal{M})^T(I - \mathcal{M})\frac{d^2}{dx^2} + Ax\frac{d}{dx}. \tag{35}$$

From (27) we have that $\|\Pi - \mathcal{M}\| < 1$ which in turn implies that A is negative definite. We use this fact to investigate the behavior of $\mathcal{L}V$ for the Lyapunov function $V(e) = \frac{1}{2}e^T e$.

Similarly as before, to ensure $V(e(t)) \leq \kappa$, we need to find a finite scalar κ and a level set $\mathcal{N}_\kappa = \{e \in \mathbb{R}^n |\, V(e) \leq \kappa\}$, such that $\mathcal{L}V(e(t)) < 0$ for all $e(t) \notin \mathcal{N}_\kappa$, where $\mathcal{L}$ is the infinitesimal generator of the process $e(t)$.

Let us first consider the SDE for the error vector $de(t) = Ae(t)dt + (I - \mathcal{M})\xi dW(t)$ and rewrite $(I - \mathcal{M})\xi dW(t) = \sum_i w_i dW_i(t)$ where

$$w_i = \xi \begin{bmatrix} -\frac{1}{N} \\ \vdots \\ 1 - \frac{1}{N} \\ \vdots \\ -\frac{1}{N} \end{bmatrix} \leftarrow i\text{th row.}$$

Then, we obtain

$$\mathcal{L}V(e) = e(t)^T Ae(t) + \frac{1}{2}\sum_{i=1}^{N} S_{ii} = e(t)^T Ae(t) + \frac{1}{2}N\xi^2[(N-1)\frac{1}{N^2} + (1 - \frac{1}{N})^2],$$

where $S = \sum_{k=1}^{N} w_k w_k^T \in \mathbb{R}^{N\times N}$, and hence $S_{ii} = \xi^2[(N-1)\frac{1}{N^2} + (1-\frac{1}{N})^2]$.

Consider now the level sets $\mathcal{N}_\kappa = \{e \in \mathbb{R}^N |\, V(e) \leq \kappa\}$ and observe that there always exists a $\hat{\kappa}$ big enough and finite such that for every $e(t) \notin \mathcal{N}_{\hat{\kappa}}$, i.e., $\frac{1}{2}e(t)^T e(t) > \hat{\kappa}$. Thus, we have $e(t)^T Ae(t) + \frac{1}{2}N\xi^2[(N-1)\frac{1}{N^2} + (1-\frac{1}{N})^2] < 0$. The latter means $\mathcal{L}V(e(t)) < 0$ for all $e(t) \notin \mathcal{N}_{\hat{\kappa}}$, which proves that every level set $\mathcal{N}_\kappa$ where $\kappa \geq \hat{\kappa}$ is contractive.

In other words, for every $e(t) \in \partial\mathcal{N}_{\hat{\kappa}}$, $e(t+dt) \in \mathcal{N}_{\hat{\kappa}}$. Equivalently, every level set $\mathcal{N}_\kappa$ where $\kappa \geq \hat{\kappa}$ is contractive. Thus, we can conclude that for every $\kappa \geq \hat{\kappa}$ there exists an $\varepsilon = \sqrt{2\kappa}$ for which the level set $\{e \in \mathbb{R}^N |\, \|e\| \leq \varepsilon\}$ is contractive. A value for $\hat{k}$ can be obtained solving the optimization problem

$$\hat{k} := \min k \quad \text{s.t.} \quad \{V(e) = k\} \subset \left\{ e(t)^T Ae(t) + \frac{1}{2}N\sigma^2[(N-1)\frac{1}{N^2} + (1 - \frac{1}{N})^2] < 0 \right\}.$$

□

4.2 Numerical Simulations

We now simulate the behavior of the most gregarious population, assuming, as in Section 4, that the hard core stubborn population is already at the equilibrium. In particular, we run three sets of simulations. The first set highlights the relationship between the system response and the coefficient ϕ_a: the mean distribution $\mu(t)$ of the most gregarious population fluctuates while converging to zero, and the variance $\sigma^2(t)$ decays gradually towards its equilibrium value.[3] The second set emphasizes how the system evolves in response to a higher coefficient ϕ_b, which corresponds to increasing the stubborn population's attraction force: both $\mu(t)$ and $\sigma^2(t)$ decrease monotonically, similarly to the evolutions shown in the first set of simulations. The third set simulates the system under various disturbance parameters ξ: they show that the variance $\sigma^2(t)$ increases with ξ.

Simulations have been run using the parameters as reported in Tables 1-2. The number of players is fixed to $N = 10^3$. The set of states is a discretization of the interval $[0, 1[$ with step size $dx = 10^{-4}$, i.e. $\mathcal{X} = \{x_{min}, x_{min} + 0.001, \dots, x_{max}\}$. The horizon length is $T = 10$, large enough to show convergence to the equilibrium distribution. As regards the initial distribution, we assume it to be Gaussian with mean $\mu_0 = 0.8$ and variance $\sigma_0^2 = 0.0025$. The parameter ξ is set to a value between 0.001 and 0.05.

[3] Being N fixed across simulations, in what follows we suppress the indicator N from the notations.

Table 1 Constant simulation parameters.

N	x_{min}	x_{max}	T	μ_0	σ_0
10^3	0	1	10	0.8	0.05

Table 2 Varying simulation parameters with different simulation sets.

	$\phi_a(0)$	$\phi_b(0)$	ξ
I	$\{1, 2, 3\}$	1.5	0.001
II	1	$\{1.5, 2.5, 3\}$	0.001
III	1	1.5	$\{0.001, 0.01, 0.05\}$

Simulation I.

The first set of simulations highlights how the coefficient that regulates the aggregation forces among the opinions, ϕ_a, is a factor in reducing the sparsity of the opinions, which is measured by the variance σ^2. From top-left to bottom-left, Figure 3 shows the distribution evolution $m(t)$ vs. the state $x(t)$ at different times. Parameter ϕ_a at time 0 varies from $\phi_a(0) = 1$ (top), $\phi_a(0) = 2$ (middle) and $\phi_a(0) = 3$ (bottom). The graph on the right displays the trajectories $\mu(t)$ (solid line and y-axis labels on the left) and $\sigma(t)$ (dashed line and y-axis labels on the right). It is worth noting that the standard deviation tends to its equilibrium value faster and faster as long as the attraction among the opinions grows: as it can be seen from the graph, the distributions at a same time instant get sharper with higher values of $\phi_a(0)$.

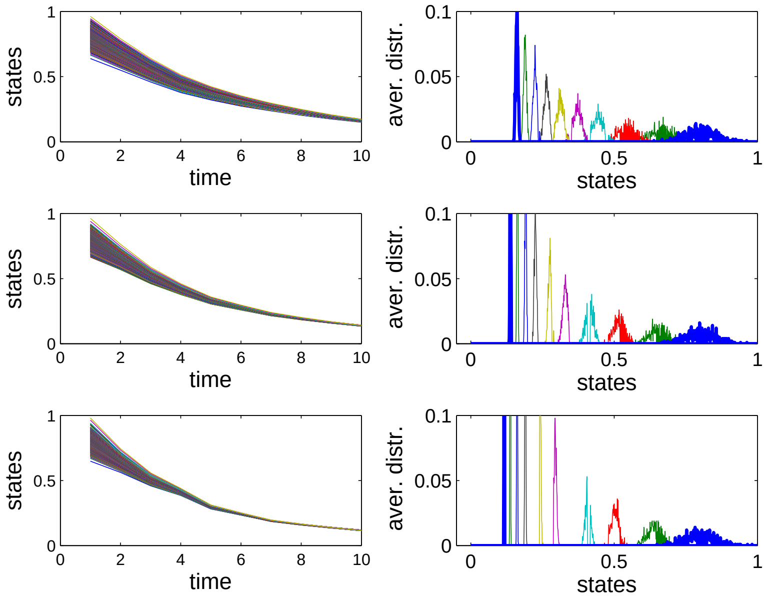

Fig. 3 Simulation I. From top to bottom, $\phi_a(0) = \{1, 2, 3\}$. The sparsity of the distribution evolution shrinks as the coefficient $\phi_a(0)$ grows. The standard deviation σ converges faster.

Simulation II.

The second set of simulations shows the connection between the coefficient $\phi_b(0)$, which describes the attracting force exhibited by the stubborn population, and the convergence speed of the mean of the most gregarious population toward zero, that is, toward the mean of the stubborn population. The graph on the left in Figure 4 shows this effect. In particular, the graph plots the time evolutions of the distribution of $x(t)$. The initial distribution is the same as in the first set of simulations, with identical initial mean and variance, while $\phi_b(0)$ varies from $\phi_b(0) = 1.5$ (top), $\phi_b(0) = 2.5$ (middle) and $\phi_b(0) = 3$ (bottom). Note that different values of $\phi_b(0)$ correspond to different values of the parameter ν_{ij} in the current cost (2) of the players of the most gregarious population. The mean opinion of the most gregarious population approaches zero with a speed that increases with $\phi_b(0)$. The graph on the right in Figure 4 displays the trajectories $\mu(t)$ (solid line and y-axis labels on the left) and $\sigma(t)$ (dashed line and y-axis labels on the right), pointing out that the mean tends to zero faster with higher values of the coefficient.

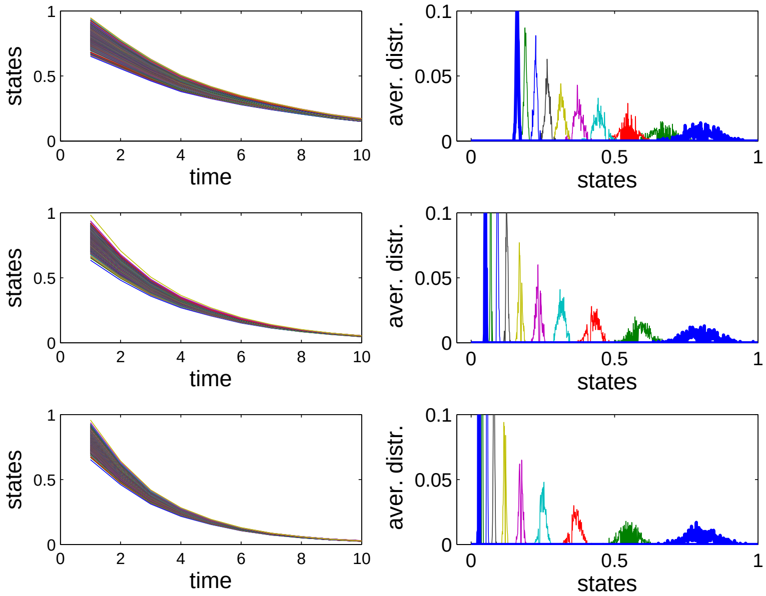

Fig. 4 Simulation II. From top to bottom, $\phi_b(0) = \{1.5, 2.5, 3\}$. The distribution converges faster to zero as $\phi_b(0)$ grows. The same for $\mu(t)$: the mean goes to zero faster.

Simulation III.

The last set of simulations shows the effects of the volatility term ξ. The initial conditions are identical to the ones of the previous simulations, and the only varying parameter is ξ from $\xi = 0.001$ (top), $\xi = 0.01$ (middle) and $\xi = 0.05$ (bottom). The graph on the left in Figure 5 shows this effect, by plotting the time evolution of the distribution of $x(t)$. Specifically one should observe the different values of the population's standard deviation as the value of ξ increases.

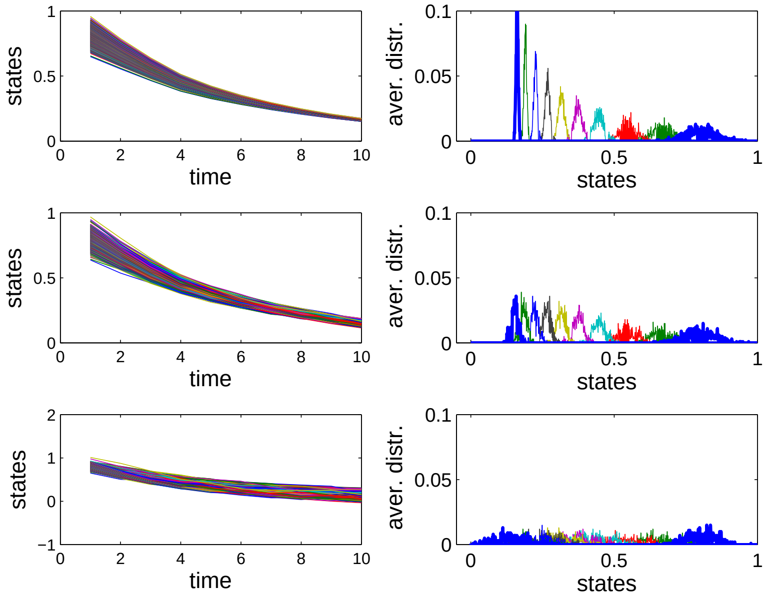

Fig. 5 Simulation III. From top to bottom, $\xi = \{0.001, 0.01, 0.05\}$. The final value of the population variance increases as the value of ξ increases.

5 Conclusions

We have considered a multi-population scenario for a mean-field game model of opinion dynamics and stubbornness. We have first analyzed conditions on the equilibrium strategies that preserve Gaussianity of the populations' densities, these being solutions of the Fokker-Planck-Kolmogorov equation. Second, we have derived explicit expressions for the optimal equilibrium strategies under the assumption that the players are myopic. Although restricted to myopic strategies, the model is rich enough to generate well recognized behavioral patterns of the macroscopic system in the context of consensus dynamics: convergence towards consensus, polarization or plurality of opinions. Our study has considered different types of populations; in particular, we have focused our attention on the interaction among two populations: one hard core and the other one most gregarious. Under this scenario, we have discussed in detail the convergence of the average behaviors in the case of a finite number of players. When the number of agents is finite, the average opinion turns out to be a stochastic process with computable bounds on first and second moments.

The authors have conducted this work as part of their current research activity on opinion dynamics. Possible extensions include the analysis of the "bandwagon effect" and the so-called "homophily". In the first case, we assume that the players want to align their opinions to the mainstream opinion, this being the opinion shared by the majority. On the other hand, homophily means that interactions occurs among players with similar opinions. Our conjecture is that mean-field game theory can successfully accommodate both phenomena.

We conclude in saying that, from our perspective, game theory offers a large range of concepts that prove to be useful in the design of information mechanisms leading to pre-specified opinion distributions. Thus the theory of “strategic thinking” can once more be a factor in the characterization of marketing strategies or advertisement campaigns.

A Appendix

The optimization problem introduced in Section 2, can be turned into a *Multi-Population Mean-Field Game*. Preliminary to the derivation of a mean-field game, is the definition of a value function as commonly done also in differential game theory or optimal control. The value function is the value of the optimization problem, carried out by each single player k, starting at time t from state x^k and for given densities $m(t)$. As we will show, the value function only depends on population's characteristics (apart from the initial state $x^k(t)$).

Proposition A.1 *Consider a generic population i and any agent k such that $i(k) = i$. Define the value function for agent k as*

$$v_i(x^k(t), t) := \sup_{u^k(\cdot)} \mathbb{E}\{\int_t^{\infty} e^{-\rho\tau} c^k(x^k(\tau), u^k(\tau), m(\tau)) d\tau)\}.$$

Then, the mean-field system is described by the equations

$$\begin{cases} \partial_t v_i(x^k(t),t) + (1-\alpha_i)\sum_{j\in I} \nu_{ij} \ln(m_j(x^k(t),t)) - \alpha_i(x^k(t) - \mu_i(0))^2 + \\ \qquad + \frac{1}{2\beta}(\partial_x v_i(x^k(t),t))^2 + \frac{\xi_i^2}{2}\partial^2_{xx} v_i(x^k(t),t) - \rho v_i(x^k(t),t) = 0, \\ \partial_t m_i(x^k(t),t) + \partial_x \Big[m_i(x^k(t),t)\Big(-\frac{1}{2\beta}\partial_x v_i(x^k(t),t)\Big)\Big] - \frac{1}{2}\xi_i^2 \partial^2_{xx} m_i(x^k(t),t) = 0, \end{cases} \tag{36}$$

for some initial population state distribution $m_i(0)$ for all $i \in I$. Furthermore, the optimal control is of the form

$$u_i^*(x^k(t),t) = -\frac{1}{2\beta}\partial_x v_i(x^k(t),t). \tag{37}$$

Proof From dynamic programming, the value function can be obtained from a corresponding maximized Hamiltonianian function H^k involving an adjoint variable p_i, called the ith co-state, and given by

$$H^k(x, p_i, m) = \sup_{u_i}\left\{ c^k(x, u_i, m) + p_i u_i \right\}.$$

From [13], the mean-field system associated to the mean-field game introduced in Section 2 is given by

$$\begin{cases} \partial_t v_i(x,t) + H^k(x, \partial_x v_i(x,t), m) + \frac{1}{2}\xi_i^2 \partial^2_{xx} v_i(x,t) - \rho v_i(x,t) = 0, \\ \partial_t m_i(x,t) + \partial_x \Big(m_i(x,t) \partial_p H^k(x, \partial_x v_i(x,t), m) \Big) - \frac{1}{2}\xi_i^2 \partial^2_{xx} m_i(x,t) = 0, \end{cases} \tag{38}$$

where $m_i(x,0) = m_{0i}(x)$ for all $i \in I$ are the initial distributions and where $x = x^k(t)$.

We first prove condition (37). To this end, let us write the Hamiltonian as:

$$H^k(x, \partial_x v_i(x,t), m) = \sup_{u_i} \Big\{ (1-\alpha_i) \sum_{j \in I} \nu_{ij} \ln(m_j(x,t)) - \alpha_i (x - \mu_{0i})^2 - \beta u^2 + \partial_x v_i(x,t) u_i \Big\} = 0. \tag{39}$$

By differentiating with respect to u_i we obtain

$$2\beta u_i(x,t) + \partial_x v_i(x,t) = 0, \tag{40}$$

which yields (37). Note that convexity of the cost functional guarantees sufficiency of the above first-order condition.

We now prove (36). Concerning the first equation, which is a PDE corresponding to the Hamilton-Jacobi-Bellman equation, let us replace u_i in the Hamiltonian (39) by its expression (37), i.e.

$$\begin{aligned} H^k(x, \partial_x v_i(x,t), m) &= (1-\alpha_i) \sum_{j \in I} \nu_{ij} \ln(m_j(x,t)) - \alpha_i (x - \mu_{0i})^2 - \beta (u_i^*(x,t))^2 + \partial_x v_i(x,t) u_i^*(x,t) \\ &= (1-\alpha_i) \sum_{j \in I} \nu_{ij} \ln(m_j(x,t)) - \alpha_i (x - \mu_{0i})^2 + \frac{1}{2\beta} (\partial_x v_i(x,t))^2. \end{aligned}$$

Using the above expression of the Hamiltonian in the first equation in (38), we obtain the Hamilton-Jacobi-Bellman equation in (36).

To prove the second equation, which is a PDE representing the Fokker-Planck-Kolmogorov equation, we simply substitute (37) into the second equation in (38), and this concludes the proof. □

The significance of the above result is that to find the optimal controls, we need to solve the set of coupled PDEs defined in (36) with given boundary conditions. This can done by iteratively solving the Hamilton-Jacobi-Bellman equation for fixed m_i and by entering the optimal u_i obtained from (37) in the Fokker-Planck-Kolmogorov equation, until a fixed point in v_i and m_i is reached [24]. In other words, it must be proved that such a map is a contraction, and to do this, we rely on compactness of the map itself and on the Schauder fixed point theorem [15]. Note that, in Proposition A.1, we do not consider a stationary control or a stationary population density distribution, although we deal with a discounted objective function over an infinite horizon. In fact, we are interested in determining the evolution of the population density distribution function over time under the general hypothesis that at time 0 the population is not distributed according to the long-term equilibrium density distribution.

A solution of (36) is said Nash-Mean Field Equilibrium as, it involves a set $\{(m_i^*, u_i^*) : i \in I\}$ of functions defined for all times $t \geq 0$ such that

$$(m_i^*, u_i^*) = \arg \sup_{m_i(.), u_i(.)} \mathbb{E}\left\{ \int_0^\infty e^{-\rho t} c^k(x^k, u_i, m) dt \Big| m_j = m_j^*, \forall j \in I \setminus \{i\} \right\} \quad \forall i \in I.$$

In other words, any player of population i does not benefit from changing its control policy u_i^* if the control policies, and therefore also the distributions, of the other populations are fixed to respectively u_j^* and m_j^* for all $j \in I \setminus \{i\}$. As a consequence, also the trajectory over time of the distribution m_i^* is unchanged.

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